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The Scaling of Higher Cumulants in a Diffusion Problem

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The scaling properties of higher cumulants for a diffusion problem are examined by means of numerical calculations. The exponent for the higher cumulants are found to be less than that of the first cumulant but larger than that of the second one. The calculations can be used for describing quantum particle diffusion in a random time-dependent potential, domain wall diffusion in a 2D magnet, etc.

KEY WORDS: Quantum particle; random time-dependent potential; anomalous diffusion; scaling for highest cumulants.

The equation

$$\frac{\partial w}{\partial t}=\frac{T}{2}\frac{\partial^2 w}{\partial x^2}+\frac{V(x, t)}{T} w \tag{1}$$

with the random uncorrelated potential $V(x, t)$,

$$\overline{V(x, t) V(x', t')}=V_0^2 \delta(x-x') \delta(t-t') \tag{2}$$

is of great importance for a wide class of problems. In biology Eq. (1) can be used for describing the variation of a population of animals due to variations in life conditions $V(x, t)$.[1] For imaginary $T=i\hbar$, Eq. (1) is the Schrödinger equation for a particle in a time-dependent random potential. Equation (1) describes also domain wall diffusion in 2D magnets with imperfections (impurities) of the "random bond" type[2-5] and directed polymers in a random medium.[6]

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Significant progress has been recently achieved in understanding the properties of the solution of Eq. (1). To be specific, we speak further in terms of the domain wall thermal fluctuations. We assume also that

$$w(x, t=0) = \delta(x) \tag{3}$$

It was shown by Huse *et al.*[2,4] and Kardar[3] that the average displacement of the domain wall is characterized by the nontrivial diffusion index

$$x_1 \equiv \left[\overline{\langle x(t)\rangle^2}\right]^{1/2} \backsim t^\alpha, \qquad \alpha = 2/3 \tag{4}$$

Here and below the angle brackets denote the thermal average.

Schulz *et al.*[5] gave important results concerning the second and higher moments of the displacements. It was shown[5] that for a number of systems, including those described by Eq. (1), the mean square of thermal fluctuation of the domain wall displacement remains the same as in the absence of the impurities ($V = 0$)

$$\begin{aligned} x_2 &\equiv \left[\overline{\langle \tilde{x}^2(t)\rangle}\right]^{1/2} \backsim t^\beta, \qquad \beta = 1/2 \\ \tilde{x} &= x - \langle x\rangle \end{aligned} \tag{5}$$

Moreover, all higher cumulants of x have the zero mean value, for example,

$$\overline{\langle \tilde{x}^4\rangle - 3\langle \tilde{x}^2\rangle^2} = 0 \tag{6}$$

Consequently, one might conclude that the solution of Eq. (1) with the condition (3) is somewhat close to a Gaussian distribution whose maximum is moving much faster than the average width is increasing [cf. Eqs. (4) and (5)]. However, such a conclusion turns out to be in drastic conflict with the numerical results obtained by Kardar and Zhang.[6] According to ref. 6, the most profitable trajectories form a treelike structure, or, in other words, the function $w(x, t)$ is in fact a set of sharp peaks. Therefore, the form of the function $w(x, t)$ differs essentially from a Gaussian one, but the difference disappears after averaging. It is obvious that this difference can be characterized by the dispersion of high-order cumulants, i.e., by quantities of the type

$$x_3 \equiv \left(\overline{\langle \tilde{x}^3(t)\rangle^2}\right)^{1/6} \backsim t^\gamma \tag{7}$$

$$x_4 \equiv \left[\overline{(\langle \tilde{x}^4(t)\rangle - 3\langle x^2(t)\rangle^2)^2}\right]^{1/8} \backsim t^\delta \tag{8}$$

The values of γ and δ are unknown; however, we may expect them to be larger than the classical diffusion index $\beta = 1/2$. The aim of the present paper is to find the values of γ and δ with the help of numerical calculations.

For this purpose we use the iteration scheme

$$w(x, t+1) = \left(1 + \frac{V(x, t)}{T} - T\right) w(x, t) + \frac{T}{2} \left[w(x+1, t) + w(x-1, t)\right] \qquad (9)$$

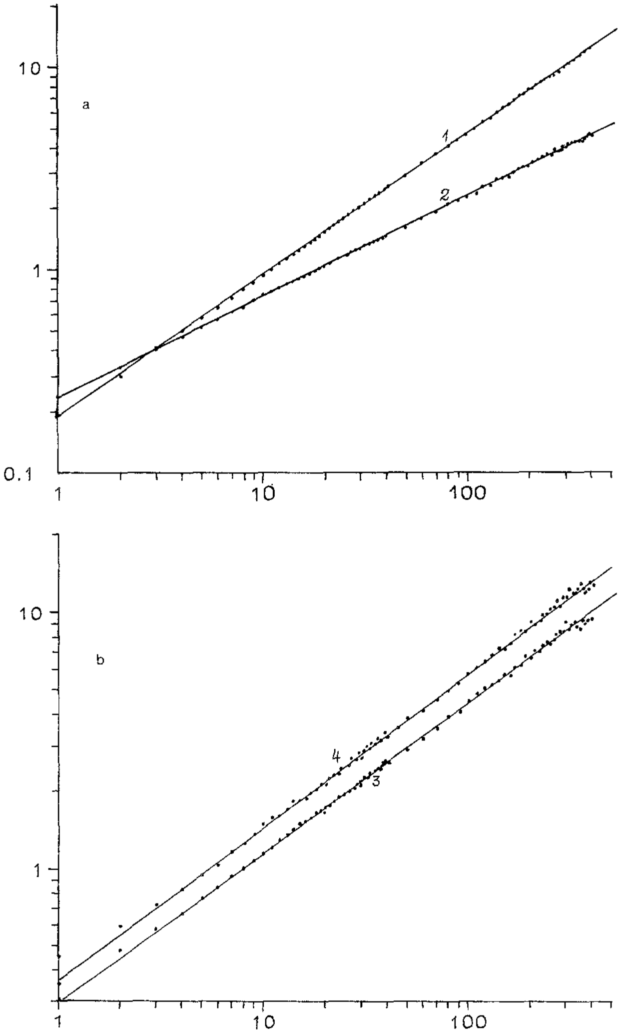

Fig. 1. Scaling properties for different cumulants for Eq. (1): (a) line 1 corresponds to $x_1(t)$, $\alpha = 0.69$; line 2 corresponds to $x_2(t)$, $\beta = 0.51$; (b) line 3 corresponds to $x_3(t)$, $\gamma = 0.60$; line 4 corresponds to $x_4(t)$, $\delta = 0.60$.

corresponding to Eq. (1), and its modified version proposed by Kardar,[3]

$$w(x, t+1) = \exp[-V(x, t)][w(x, t) + \gamma w(x+1, t) + \gamma w(x-1, t)] \quad (10)$$

The results of our calculations by means of the Kardar iteration (10) are shown in Fig. 1. In the calculations we used approximately the same values of parameters as Kadar[3] did: $V_0 = 1$, $\gamma = 0.1$. Figure 1a demonstrates the scaling properties of first (x_1) and second (x_2) cumulants defined by Eqs. (4) and (5), respectively. The values obtained, $\alpha = 0.69$ and $\beta = 0.51$, are in good agreement with the results adduced in refs. 2–6. Figure 1b shows the results for third (x_3) and fourth (x_4) cumulants defined by Eqs. (7) and (8): $\gamma = \delta = 0.60$. One finds the same values by using the iteration procedure described by Eq. (9), though the range of intermediate processes for small t is wider in the latter case.

Comparing Figs. 1a and 1b, one can see that the higher cumulants x_3 and x_4 possess stronger fluctuations from sample to sample than x_1 and x_2. For this reason, when obtaining the curves displayed in Fig. 1, we have to average over 6000 realizations for $V(x, t)$, not over 400 as in Ref. 3. Performing specific verifications, we confirmed the results to be identical in the three following cases: (i) the coefficient T is real, and $w(x, t)$ is proportional to the possibility of getting to the point (x, t) from the point $(0, 0)$; (ii) T is real, and the possibility is proportional to w^2; (iii) T is imaginary, and the possibility is proportional to ww^* (see also ref. 7).

Thus, if $V(x, t)$ is nonzero, the statistics of the domain wall thermal fluctuations differs considerably from that for a pure crystal. Nevertheless, the mean square of the fluctuation is the same for both impure and pure crystals and the main values of higher cumulants vanish; their dispersions governed by Eqs. (7) and (8) have stronger divergences than the second moment defined by Eq. (5). The values of scaling indices for x_3 and x_4 appeared to be identical; therefore, it may be possible that the same index characterizes all the higher cumulants. At the same time, the index of highest cumulants appeared to be smaller than that of the first-order one, which is why most of the physical properties of the domain wall must be driven by the first-moment index. This concerns other systems described by Eq. (1) as well.

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